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On transverse vibrations of a vertical Timoshenko beam

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Abstract

In this paper the transverse vibrations of a standing, uniform Timoshenko beam will be considered. Due to gravity and the self-weight of the beam a linearly varying compression force is acting on the beam. It will be assumed that this compression force is small but not negligible. The transverse vibrations of the beam can be described by an initial-boundary value problem. Approximations of the solution of this problem will be constructed by using a multiple time-scales perturbation method. Also approximations of the frequencies will be obtained. Moreover, the effect of the linearly varying compression force on the magnitude of the frequencies of the oscillation modes of the beam will be discussed.

1. Introduction

Many structures, such as bridges, buildings, and spacecraft arms can be modelled as flexible beams. The vibrations of a bridge can be modelled as a horizontal beam. In Ref. [1] a horizontal beam has been considered as a model for a bridge. The vibrations of a tall building can be modelled as a vertical beam. A vertical beam in a gravity field is subjected to an axial force due to the self-weight of the beam. A standing beam is subjected to a compressive axial force and a hanging beam to a tensile axial force. An example of a standing beam is a tall building and an example of a hanging beam is a stiff elevator cable. The theory of Euler–Bernoulli and Timoshenko can be used to describe the vibrations of a beam. The model that describes the transverse vibrations of a vertical beam, due to the bending moment only, is the Euler–Bernoulli beam theory. This theory is not sufficient for short beams or for the higher modes of slender beams because of ignoring the shear force and the rotatory moment of inertia. The Timoshenko beam theory includes the effects of shear force and rotatory inertia.

In Ref. [2] the mode shape differential equation describing the transverse vibrations of a hanging Euler–Bernoulli beam under linearly varying axial force has been derived. It has been concluded that the equation can not be solved exactly. In Ref. [2] approximate analytical solutions have been determined by using the Ritz–Galerkin method with gravity-free eigenfunctions. Moreover, in Ref. [2], approximate analytical solutions of this problem have been determined for the case that gravity is dominating by using the method of

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matching asymptotic expansions. In Ref. [3] this method have been applied to a similar problem, that is, to the problem of a slightly stiff pendulum carrying a small bob. Also it has been shown in Ref. [2] that a compression force reduces the frequencies and that the influence of the gravity on the frequencies decreases by increasing mode number. In Ref. [4] the natural frequencies of standing and hanging Euler–Bernoulli beams have been studied. The Frobenius method has been used to solve the mode shape differential equation of a uniform hanging beam. It has been concluded in Ref. [4] that the natural frequencies of the hanging and of the standing beam are noticeably different. In Ref. [5] buckling of an Euler–Bernoulli beam under self-weight has been studied. In Refs. [6,7] the partial differential equation describing the vibrations of a standing Euler–Bernoulli beam with tip-mass has been derived. A multiple time-scales perturbation method has been used to solve this problem for the case that the influence of the axial load is small. It has been concluded in Refs. [6,7] that increasing the gravity effect (i.e. increasing compression force) and increasing the mass of the tip-mass reduces the natural frequencies. In Ref. [8] Hamilton's principle has been used to obtain the governing equations of a vertically hanging Timoshenko beam under gravity as a model for flexible space structures. The study in Ref. [8] is restricted to hanging beams, since standing beams under dominating gravity load will buckle due to its own weight. In Ref. [8], by using a finite element approach, the vibrational behaviour of the flexible beam has been determined. It has been shown that the frequencies of the vibration modes of the beam increase with increasing gravity effect and that the influence of the gravity on these frequencies decreases with increasing mode number. Also it has been concluded in Ref. [8] that the inclusion of shear deformation and rotatory inertia reduces the increases (due to the tensile axial force which is acting on the beam) of the frequencies in the higher order modes of the hanging beam. These results have also been found in Ref. [9], where the vibrations of a hanging Timoshenko beam have been studied by using the Galerkin method. In Ref. [10] uniform and non-uniform beams with various types of boundary conditions and with axial force have been studied. And in Ref. [11] the transverse buckling of a rotating Timoshenko beam have been studied for clamped–free and clamped–clamped boundary conditions.

In this paper the vibrations of a standing, uniform, cantilevered beam as a simple model for a tall building will be studied. The beam is subjected to a linearly varying compression force. Inclusion of this compression force into the beam model reduces the magnitude of the frequencies of the beam. The aim of this paper is to examine this decrease in magnitude of the frequencies, more precisely, to study the influence of the beam parameters on this decrease. It will be assumed that the compression force due to gravity is small but not negligible. The Timoshenko beam theory will be used to model the transverse vibrations of the beam. Now the vibrations can be described by an initial-boundary value problem. The multiple time-scales perturbation method will be used to obtain explicit approximations of the solutions of this initial-boundary value problem. Moreover, explicit approximations of the natural frequencies will be obtained. Note that the methods used in this paper are not restricted to standing beams, but can also be applied to hanging beams. This is the case of a beam under linearly varying tensile force.

This paper is organized as follows. Firstly, in Section 2, the governing partial differential equations describing the transverse vibrations of a standing, uniform, cantilevered Timoshenko beam will be derived. Secondly, in Section 3, the eigenvalue problem of a standing, uniform, cantilevered Timoshenko beam will be derived. It will be shown that the eigenfunctions form an orthogonal set and that the eigenvalues are real-valued and positive for sufficient small gravity effect. Then, in Section 4, the gravity effect will be neglected. The initial-boundary value problem describing the transverse vibration of a uniform, cantilevered Timoshenko beam will be solved exactly. In Section 5 the partial differential equations describing the vibrations of a standing, uniform, cantilevered Timoshenko beam will be solved approximately by using a multiple time-scales perturbation method. Also the effect of gravity on the frequencies and the oscillation modes will be derived. Finally, in Section 6 conclusions will be drawn and remarks will be made.

2. Equations of motion

In this section the linearized equations of motion that describe the transverse vibration and the rotation of the cross-section of a vertical, uniform, cantilevered beam (see Fig. 1) will be derived by using the Bress–Timoshenko beam theory and the classical dynamic equilibrium method. Due to gravity and due to the self-weight of the beam a linearly varying axial compression force is acting on the beam. To describe the effect

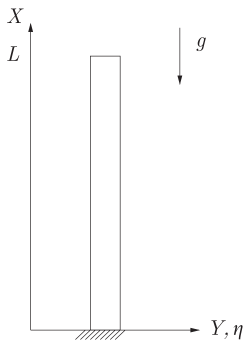

Fig. 1. A simple model of a standing, cantilevered beam.

of the axial force it is assumed in this paper that the axial force is tangential to the slope of the beam. It can also be assumed that the axial force is normal to the direction of the shear force. In Ref. [12] both cases, the axial force is tangential to the axis of the slope of the beam and the axial force is normal to the shearing force, have been considered and for both cases the equations of motion have been derived. But in Ref. [12] nothing has been said on which method is more accurate. However, in Ref. [13] it has been indicated that the equations of motion which follows from the first assumption are more accurate. Therefore, also in this paper it is assumed that the axial force is tangential to the slope of the beam. Note that also in Refs. [10,14] both cases have been considered. For Bress–Timoshenko beam theory the total slope of the beam, the bending moment, and the shearing force are given by (see Ref. [15])

$$\frac{\partial \eta(X,\tau)}{\partial X} = \psi(X,\tau) + \beta(X,\tau), \tag{1}$$

$$M(X,\tau) = EI\frac{\partial \psi(X,\tau)}{\partial X}, \tag{2}$$

$$V(X,\tau) = -k'\beta(X,\tau)AG = -k'AG\left(\frac{\partial \eta(X,\tau)}{\partial X} - \psi(X,\tau)\right), \tag{3}$$

respectively, where $M(X,\tau)$ is the moment, $V(X,\tau)$ is the shear force, E is the Young modulus, I is the moment of inertia of the cross-section, k' is the shear coefficient depending on the shape of the cross-section, G is the modulus of elasticity in shear or the modulus of rigidity, A is the cross-sectional area, $\eta(X,\tau)$ is the deflection of the beam in Y-direction (see Fig. 2), $\psi(X,\tau)$ is the cross-sectional rotation angle due to bending, $\beta(X,\tau)$ is the shear angle, τ is the time, and X is the position along the beam. From the Timoshenko beam element (see Fig. 2) a dynamic equilibrium for the forces in Y-direction and the moments about point n acting on this beam element can be obtained. The angles $\psi(X+\mathrm{d}X,\tau)$ and $\psi(X,\tau)$, and the slopes $\partial\eta(X+\mathrm{d}X,\tau)/\partial X$ and $\partial\eta(X,\tau)/\partial X$ are assumed to be small. By linearizing the so-obtained equilibria with respect to $\psi(X+\mathrm{d}X,\tau)$, $\psi(X,\tau)$, $\partial\eta(X+\mathrm{d}X,\tau)/\partial X$, and $\partial\eta(X,\tau)/\partial X$ it follows that the equilibrium for the forces is approximately given by

$$\begin{aligned} V(X,\tau) - V(X+\mathrm{d}X,\tau) - \rho A\,\mathrm{d}X\frac{\partial^2\eta(X+\mathrm{d}X/2,\tau)}{\partial\tau^2} + S(X)\frac{\partial\eta(X,\tau)}{\partial X} \\ - S(X+\mathrm{d}X)\frac{\partial\eta(X+\mathrm{d}X,\tau)}{\partial X} = 0 \end{aligned} \tag{4}$$

and that the equilibrium for the moments is approximately given by

$$\begin{aligned} M(X,\tau) - M(X+\mathrm{d}X,\tau) + V(X,\tau)\,\mathrm{d}X - \rho A\frac{(\mathrm{d}X)^2}{2}\frac{\partial^2\eta(X+\mathrm{d}X/2,\tau)}{\partial\tau^2} \\ + \rho I\,\mathrm{d}X\frac{\partial^2\psi(X+\mathrm{d}X/2,\tau)}{\partial\tau^2} = 0, \end{aligned} \tag{5}$$

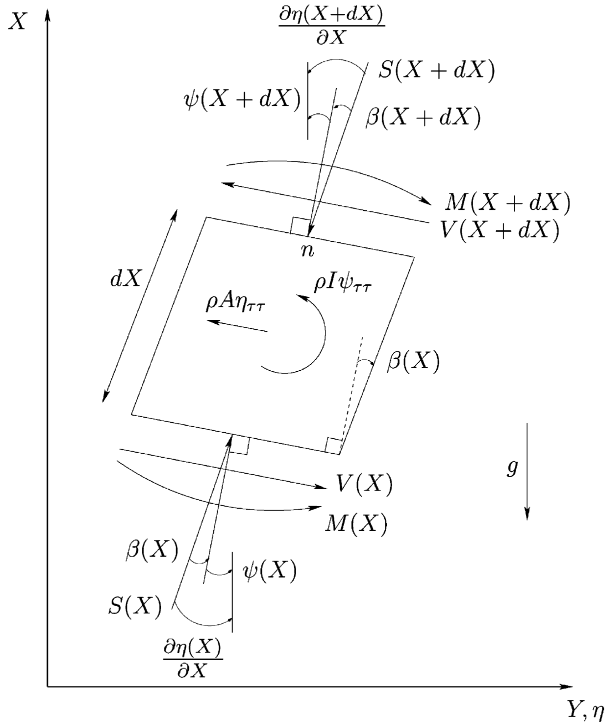

Fig. 2. Timoshenko beam element.

where $S(X) = g\rho A(L - X)$, g is the acceleration due to gravity, L the length of the beam, and ρ is the density of the beam. Now substitute the Taylor series of $V(X + \mathrm{d}X)$ about X into Eq. (4), and substitute the Taylor series of $M(X + \mathrm{d}X)$ about X into Eq. (5). Then divide the so-obtained equations by $\mathrm{d}X$, and take the limit $\mathrm{d}X \to 0$, to obtain the following equations:

$$\frac{\partial V(X,\tau)}{\partial X} + \rho A \frac{\partial^2 \eta(X,\tau)}{\partial \tau^2} + \frac{\partial}{\partial X}\left(S(X)\frac{\partial \eta(X,\tau)}{\partial X}\right) = 0, \tag{6}$$

$$\frac{\partial M(X,\tau)}{\partial X} - V(X,\tau) - \rho I \frac{\partial^2 \psi(X,\tau)}{\partial \tau^2} = 0. \tag{7}$$

The boundary conditions of a cantilevered beam are given by

$$\eta(0,\tau) = \psi(0,\tau) = 0, \tag{8}$$

$$M(L,\tau) = V(L,\tau) = 0. \tag{9}$$

By substituting Eqs. (2) and (3) into Eqs. (6)–(9) the following coupled partial differential equations and boundary conditions describing the deflection and the angle of rotation of a uniform, cantilevered Timoshenko beam are obtained:

$$k'AG(\eta_{XX} - \psi_X) - \rho A \eta_{\tau\tau} - g\rho A[(L - X)\eta_X]_X = 0, \tag{10}$$

$$EI\psi_{XX} + k'AG(\eta_X - \psi) - \rho I \psi_{\tau\tau} = 0, \tag{11}$$

$$\eta(0,\tau) = \psi(0,\tau) = 0, \tag{12}$$

$$EI\psi_X(L,\tau) = 0, \tag{13}$$

$$k'AG(\eta_X(L,\tau)-\psi(L,\tau))=0. \tag{14}$$

To put the equations of motion (10)–(14) in a non-dimensional form the following substitutions $x=X/L$, $u=\eta/L$, and $t=\kappa\tau$, where $\kappa=1/L^2\sqrt{EI/\rho A}$, will be used. By applying these substitutions to Eqs. (10)–(14) the following initial-boundary value problem is obtained:

$$\psi_{xx}+\left(\frac{1}{r^2s^2}\right)(u_x-\psi)-r^2\psi_{tt}=0,\quad 0<x<1,\quad t>0, \tag{15}$$

$$\left(\frac{1}{r^2s^2}\right)(u_{xx}-\psi_x)-u_{tt}-\varepsilon[\tilde{S}(x)u_x]_x=0,\quad 0<x<1,\quad t>0, \tag{16}$$

$$u(0,t)=\psi(0,t)=0,\quad t\geqslant 0, \tag{17}$$

$$\psi_x(1,t)=0,\quad t\geqslant 0, \tag{18}$$

$$u_x(1,t)-\psi(1,t)=0,\quad t\geqslant 0, \tag{19}$$

$$u(x,0)=f(x)\quad\text{and}\quad u_t(x,0)=h(x),\quad 0<x<1, \tag{20}$$

$$\psi(x,0)=p(x)\quad\text{and}\quad \psi_t(x,0)=q(x),\quad 0<x<1, \tag{21}$$

where $\tilde{S}(x)=1-x$, $r^2=I/AL^2, s^2=E/k'G$, and $\varepsilon=g\rho AL^3/EI$, and where $f(x)$, $h(x)$, $p(x)$, and $q(x)$ are the initial displacement of the beam in horizontal direction at position x, the initial velocity of the beam in horizontal direction at position x, the initial rotation angle (due to bending) at position x, and the initial angular velocity at position x, respectively. It should be observed that ε, r^2 and s^2 are dimensionless parameters. The parameter ε is the gravity parameter, which may be regarded as the ratio of the weight multiplied by the square of the length to the flexural rigidity (see also Ref. [8]). Note that from Eqs. (15)–(21) the equations of motion, which describes the vibrations of a hanging beam, can be obtained by assuming that the gravity force acts in opposite direction. In this paper it will be assumed that the gravity parameter is small, that is, $0<\varepsilon\ll 1$. The parameter $1/r$ is the slenderness ratio and $1/rs$ is the shear/flexural rigidity ratio. The parameters r^2 and s^2 are assumed to be ε-independent. In this paper the effect of the parameters ε, r^2, and s^2 on the frequencies will be studied. Note that by eliminating ψ from Eqs. (15)–(21) an initial-boundary value problem for u can be obtained. By substituting $s=0$ into the so-obtained problem the problem that describes the transverse vibrations of a Rayleigh beam can obtained. If additionally $r=0$ is substituted into this problem the equations of motion of a cantilevered Euler–Bernoulli beam are obtained.

3. A perturbation method

In this section the initial-boundary value problem (15)–(19) will be considered. This problem describes the transverse vibrations of a standing Timoshenko beam. Now look for non-trivial solutions of the system (15)–(19) in the form $u(x,t)=U(x)T_1(t)$ and $\psi(x,t)=\hat{\Psi}(x)T_2(t)$. Note that $u(x,t)\equiv 0$ only leads to $\psi(x,t)\equiv 0$ and that $\psi(x,t)\equiv 0$ only leads to $u(x,t)\equiv 0$. By substituting $u(x,t)=U(x)T_1(t)$ and $\psi(x,t)=\hat{\Psi}(x)T_2(t)$ into Eq. (15) it follows that

$$(\hat{\Psi}(x)-r^2s^2\hat{\Psi}''(x))T_2(t)+r^4s^2\hat{\Psi}(x)T_2''(t)=U'(x)T_1(t), \tag{22}$$

where the primes denote differentiation with respect to the independent variable, whether x or t. Let $c_1,c_2\in\mathbb{C}$. From Eq. (22) it follows that the case $T_1(t)\neq c_1T_2(t)$ and $T_1(t)\neq c_2T_2''(t)$ leads to $U'(x)\equiv 0$. Hence, from Eq. (17), it follows that $U(x)\equiv 0$. Therefore, following the argument as given above Eq. (22), it follows that the case $T_1(t)\neq c_1T_2(t)$ and $T_1(t)\neq c_2T_2''(t)$ only leads to trivial solutions. If $T_1(t)\neq c_1T_2(t)$ and $T_1(t)=c_2T_2''(t)$ it follows from Eqs. (17)–(19) and (22) that $\hat{\Psi}-r^2s^2\hat{\Psi}''=0$ and $\hat{\Psi}(0)=\hat{\Psi}(1)=\hat{\Psi}'(1)=0$. Hence also this case only leads to trivial solutions. Therefore, from Eq. (22), the case $T_1(t)\neq c_1T_2(t)$ and $T_1(t)\neq c_2T_2''(t)$, and the case $T_1(t)\neq c_1T_2(t)$ and $T_1(t)=c_2T_2''(t)$ it follows that Eqs. (15)–(21) can only have non-trivial solutions if there exists a constant $c_1\in\mathbb{C}\backslash\{0\}$ such that $T_1(t)=c_1T_2(t)$. Now, look for non-trivial solutions of systems (15)–(19) in the form $u(x,t)=U(x)T(t)$ and $\psi(x,t)=\Psi(x)T(t)$, where $c_1\Psi(x)=\hat{\Psi}(x)$. By substituting this into

Eq. (15) it follows that

$$\frac{T''}{T}=\frac{U'(x)-(\Psi(x)-r^2s^2\Psi''(x))}{r^4s^2\Psi(x)}=-\lambda, \tag{23}$$

where $\lambda\in\mathbb{C}$ is a complex-valued separation constant. Now, substitute $u(x,t)=U(x)T(t)$, $\psi(x,t)=\Psi(x)T(t)$, and $T''=-\lambda T$ into Eqs. (15)–(19) to obtain the following eigenvalue problem:

$$\Psi''+\frac{1}{r^2s^2}(U'-\Psi)=-r^2\lambda\Psi, \tag{24}$$

$$\frac{1}{r^2s^2}(U''-\Psi')-\varepsilon[(1-x)U']'=-\lambda U, \tag{25}$$

$$\Psi(0)=U(0)=0, \tag{26}$$

$$\Psi'(1)=0, \tag{27}$$

$$U'(1)-\Psi(1)=0. \tag{28}$$

The eigenvalue λ corresponds to the eigenfunction $\mathbf{\Phi}(x)$ defined by

$$\mathbf{\Phi}(x)=\begin{pmatrix}U\\ \Psi\end{pmatrix}. \tag{29}$$

Multiply the left-hand sides of Eqs. (24) and (25) by the non-trivial functions $\overline{\Psi(x)}$ and $\overline{U(x)}$, respectively, sum these so-obtained expressions, and integrate the so-obtained sum by parts with respect to x from 0 to 1, to obtain

$$\begin{aligned}&\int_0^1\left\{\left(\Psi''+\frac{1}{r^2s^2}(U'-\Psi)\right)\overline{\Psi}+\frac{1}{r^2s^2}(U''-\Psi'-\varepsilon r^2s^2[(1-x)U']')\overline{U}\right\}\mathrm{d}x\\ &=\int_0^1\left\{\overline{\left(\Psi''+\frac{1}{r^2s^2}(U'-\Psi)\right)}\Psi+\frac{1}{r^2s^2}\overline{(U''-\Psi'-\varepsilon r^2s^2[(1-x)U']')}U\right\}\mathrm{d}x.\end{aligned} \tag{30}$$

Now, substitute Eqs. (24) and (25) into Eq. (30) to obtain

$$(\lambda-\overline{\lambda})\int_0^1\{U(x)\overline{U(x)}+r^2\Psi(x)\overline{\Psi(x)}\}\,\mathrm{d}x=0. \tag{31}$$

Since $U\overline{U}=|U|^2\geqslant 0$, $\Psi\overline{\Psi}=|\Psi|^2\geqslant 0$, and because the functions $U(x)$ and $\Psi(x)$ are not allowed to be identically equal to zero the integrand in Eq. (31) is positive. Therefore $\lambda-\overline{\lambda}=0$, which implies that λ is real. Since the eigenvalues λ, and the parameters (r^2, s^2, and ε) in the differential Eqs. (24) and (25), and the boundary conditions (26) and (28) are real-valued, it follows that the eigenfunction $\mathbf{\Phi}(x)$ can be chosen to be real-valued. Let the vector function $\mathbf{\Phi}_i(x)$ be a vector solution of Eqs. (24)–(28) corresponding to the eigenvalue λ_i and let $\mathbf{\Phi}_j(x)$ be a vector solution of Eqs. (24)–(28) corresponding to the eigenvalue λ_j. Then, again by using integration by parts it follows that

$$(\lambda_i-\lambda_j)\int_0^1\{U_iU_j+r^2\Psi_i\Psi_j\}\,\mathrm{d}x=0. \tag{32}$$

Hence $\int_0^1\{U_iU_j+r^2\Psi_i\Psi_j\}\,\mathrm{d}x=0$ if $\lambda_i\neq\lambda_j$. So, eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the inner product defined by

$$\langle\mathbf{\Phi}_i,\mathbf{\Phi}_j\rangle=\int_0^1\{U_iU_j+r^2\Psi_i\Psi_j\}\,\mathrm{d}x. \tag{33}$$

Now it will be shown that the eigenvalues are positive for sufficiently small values of ε. Multiply Eq. (24) by $\Psi(x)$, multiply Eq. (25) by $U(x)$, sum the so-obtained results, and integrate the so-obtained sum with respect to x from 0 to 1, to obtain

$$\int_0^1 \left\{ (\Psi'(x))^2 + \frac{1}{r^2 s^2}(U'(x) - \Psi(x))^2 - \varepsilon(1-x)(U'(x))^2 \right\} \mathrm{d}x = \lambda \int_0^1 \{U^2(x) + r^2 \Psi^2(x)\} \, \mathrm{d}x. \tag{34}$$

It should be observed that the integral at the right-hand side of Eq. (34) is positive. Now it will be shown that the left-hand side of Eq. (34) is positive for sufficiently small values of ε. Then it can be concluded that the eigenvalues are positive for sufficiently small values of ε. Using $\Psi(x) = \int_0^x \Psi'(s) \, \mathrm{d}s$ the following inequality on $0 < x < 1$ can be derived:

$$|\Psi(x)| \leqslant \int_0^x |\Psi'(s)| \, \mathrm{d}s \leqslant \int_0^1 |\Psi'(x)| \, \mathrm{d}x. \tag{35}$$

Using the Cauchy–Schwarz inequality it follows that

$$(\Psi(x))^2 \leqslant \left(\int_0^1 |\Psi'(x)| \, \mathrm{d}x \right)^2 \leqslant \int_0^1 (\Psi'(x))^2 \, \mathrm{d}x. \tag{36}$$

Furthermore, it should be observed that from $(U'(x) - (1 + 2r^2 s^2)\Psi(x))^2 = (1 + 2r^2 s^2)(U'(x) - \Psi(x))^2 + 2r^2 s^2 (1 + 2r^2 s^2)\Psi^2(x) - 2r^2 s^2 (U'(x))^2$ it follows that

$$(U'(x))^2 \leqslant \left(\frac{1 + 2r^2 s^2}{2r^2 s^2} \right) (U'(x) - \Psi(x))^2 + (1 + 2r^2 s^2)\Psi^2(x). \tag{37}$$

Substitution of Eq. (37) into the left-hand side of Eq. (34) yields

$$\begin{aligned} &\int_0^1 \left\{ (\Psi'(x))^2 + \frac{1}{r^2 s^2}(U'(x) - \Psi(x))^2 - \varepsilon(1-x)(U'(x))^2 \right\} \mathrm{d}x \\ &\geqslant \int_0^1 \{(\Psi'(x))^2 - \varepsilon(1 + 2r^2 s^2)(1-x)\Psi^2(x)\} \, \mathrm{d}x \\ &\quad + \frac{1}{r^2 s^2} \int_0^1 \left\{ \left(1 - \frac{\varepsilon}{2}(1 + 2r^2 s^2)(1-x)\right)(U'(x) - \Psi(x))^2 \right\} \mathrm{d}x. \end{aligned} \tag{38}$$

Then, by using the inequality $\int_0^1 \{(1-x)(U'(x) - \Psi(x))^2\} \, \mathrm{d}x \leqslant \int_0^1 (U'(x) - \Psi(x))^2 \, \mathrm{d}x$ and inequality (36), it follows that inequality (38) leads to

$$\begin{aligned} &\int_0^1 \left\{ (\Psi'(x))^2 + \frac{1}{r^2 s^2}(U'(x) - \Psi(x))^2 - \varepsilon(1-x)(U'(x))^2 \right\} \mathrm{d}x \\ &\geqslant \left(1 - \varepsilon(1 + 2r^2 s^2)\left(\int_0^1 (1-x) \, \mathrm{d}x\right)\right)\left(\int_0^1 (\Psi'(x))^2 \, \mathrm{d}x\right) \\ &\quad + \frac{1}{r^2 s^2} \int_0^1 \left\{ \left(1 - \frac{\varepsilon}{2}(1 + 2r^2 s^2)\right)(U'(x) - \Psi(x))^2 \right\} \mathrm{d}x \\ &= \left(1 - \frac{\varepsilon}{2}(1 + 2r^2 s^2)\right)\left(\int_0^1 \left\{ (\Psi'(x))^2 + \frac{1}{r^2 s^2}(U'(x) - \Psi(x))^2 \right\} \mathrm{d}x\right). \end{aligned} \tag{39}$$

Hence from Eq. (34), inequality (39), and since $\Psi'(x))^2 + (1/r^2 s^2)(U'(x) - \Psi(x))^2 \equiv 0$ only leads to trivial solutions, it follows that the eigenvalues are certainly positive if

$$\varepsilon < \frac{2}{1 + 2r^2 s^2}. \tag{40}$$

It will be assumed that the gravity parameter, ε, is a small parameter, that is, $0 < \varepsilon \ll 1$. For this case the eigenvalues will be positive.

By eliminating ψ from Eqs. (15)–(21) an initial-boundary value problem for u can be obtained. By substituting $r = s = 0$ into the so-obtained problem the equations of motion of a cantilevered Euler–Bernoulli beam are obtained. Hence, from Eq. (40), it follows that the eigenvalues of a standing, cantilever Euler–Bernoulli beam are certainly positive if $\varepsilon < 2$. Note that this result also has been found in Ref. [6].

Although it has been shown that the eigenvalues (λ_n) are real-valued and positive for all sufficiently small values of ε, and that the corresponding eigenfunctions ($\mathbf{\Phi}_n$) can be chosen to be real-valued and are orthogonal with respect to the inner product (33), systems (15)–(19) cannot be solved exactly. Systems (15)–(19) cannot be solved exactly because of the linearly varying axial compression force acting on the beam. In this paper a multiple time-scales perturbation method will be applied to solve problem (15)–(19) approximately. The reader is referred to the book of Nayfeh and Mook [16] for a description of this method. In Section 4 the case $\varepsilon = 0$ will be considered first, and in Section 5 problems (15)–(21) with ε sufficiently small will be solved approximately.

4. The case without gravity ($\varepsilon = 0$)

In this section the transverse vibrations of a Timoshenko beam will be considered. The gravity effect is neglected. These vibrations can be described by Eqs. (15)–(21) with $\varepsilon = 0$. In the previous section it has been shown that the separated solutions of the initial-boundary value problem (15)–(19) can be found, that is, solutions $u(x,t)$ in the form $U(x)T(t)$, and solutions $\psi(x,t)$ in the form $\Psi(x)T(t)$, where $T'' + \lambda T = 0$, and where $\lambda \in \mathbb{C}$ is a separation constant. Now, by substituting this into Eqs. (15)–(19) with $\varepsilon = 0$ the following problem is obtained:

$$\Psi'' + \frac{1}{r^2 s^2}(U' - \Psi) = -r^2 \lambda \Psi, \tag{41}$$

$$\frac{1}{r^2 s^2}(U'' - \Psi') = -\lambda U, \tag{42}$$

$$\Psi(0) = U(0) = 0, \tag{43}$$

$$\Psi'(1) = 0, \tag{44}$$

$$U'(1) - \Psi(1) = 0. \tag{45}$$

In Ref. [17] this problem has been studied for the case $r^4 s^2 \lambda \neq 1$. In Ref. [17], for the case $r^4 s^2 \lambda \neq 1$, a so-called characteristic equation (also called a frequency equation) and equations for the eigenfunctions corresponding to simple eigenvalues have been obtained. In this section the case $r^4 s^2 \lambda \neq 1$ and the case $r^4 s^2 \lambda = 1$ will be discussed. For the case $r^4 s^2 \lambda \neq 1$ the characteristic equation will be obtained. Furthermore, it will be shown that an eigenvalue of problem (41)–(45) can have two independent eigenfunctions, such an eigenvalue is called a double eigenvalue (see Ref. [18]). Moreover, it will be shown that Eqs. (41)–(45) can only have such a double eigenvalue if $s^2 = 1$. For the case $r^4 s^2 \lambda = 1$ it will be shown that double eigenvalues do not exist. Furthermore, it will be shown that $\lambda = 1/r^4 s^2$ is only an eigenvalue for specific values of the parameters r and s. Also the eigenfunctions for the case $r^4 s^2 \lambda = 1$ will be obtained. Next, in this section, the solution of the initial-boundary value problem (15)–(21) with $\varepsilon = 0$ will be given. Lastly, approximate forms of the eigenvalues will be derived.

Firstly the case $r^4 s^2 \lambda \neq 1$ will be studied. If $r^4 s^2 \lambda \neq 1$ the solution of Eqs. (41)–(42) can be given by

$$\hat{\mathbf{\Phi}}(x) = \begin{pmatrix} U \\ \Psi \end{pmatrix}, \tag{46}$$

where

$$U(x) = c_0 \cosh(\omega_1 x) + c_1 \sinh(\omega_1 x) + c_2 \cos(\omega_2 x) + c_3 \sin(\omega_2 x), \tag{47}$$

$$\Psi(x) = d_0 \cosh(\omega_1 x) + d_1 \sinh(\omega_1 x) + d_2 \cos(\omega_2 x) + d_3 \sin(\omega_2 x), \tag{48}$$

where

$$\omega_{1,2} = \sqrt{\frac{r^2 \lambda}{2}} \sqrt{\mp(1 + s^2) + \sqrt{(1 - s^2)^2 + \frac{4}{r^4 \lambda}}}, \tag{49}$$

and where the constants c_i and d_i, where $i = 0, 1, 2, 3$, in Eqs. (47) and (48) are unknown so far. Note that, in the previous section, it has been shown that the eigenvalue λ is real-valued and positive. From Eq. (42) it follows that the constants c_i depend on d_i in the following way:

$$(\omega_1^2 + r^2 s^2 \lambda) c_0 = \omega_1 d_1, \tag{50}$$

$$(\omega_1^2 + r^2 s^2 \lambda) c_1 = \omega_1 d_0, \tag{51}$$

$$(r^2 s^2 \lambda - \omega_2^2) c_2 = \omega_2 d_3, \tag{52}$$

$$(\omega_2^2 - r^2 s^2 \lambda) c_3 = \omega_2 d_2. \tag{53}$$

By using Eq. (41) similar relations between c_i and d_i can be found. Now, from the boundary conditions (43)–(45), it follows that a solution of problems (41)–(45) can only exist if $\mathbf{Ad} = \mathbf{0}$, where $\mathbf{d} = [d_0, d_3, d_2, d_1]^{\mathrm{T}}$, and where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \dfrac{\omega_1}{\omega_2}\zeta \\ \omega_1 \sinh(\omega_1) & \omega_2 \cos(\omega_2) & -\omega_2 \sin(\omega_2) & \omega_1 \cosh(\omega_1) \\ \zeta \cosh(\omega_1) & \sin(\omega_2) & \cos(\omega_2) & \zeta \sinh(\omega_1) \end{pmatrix}, \tag{54}$$

where

$$\zeta = \frac{r^2 s^2 \lambda - \omega_2^2}{\omega_1^2 + r^2 s^2 \lambda} = -\frac{\omega_1^2 + r^2 \lambda}{\omega_1^2 + r^2 s^2 \lambda}. \tag{55}$$

By elementary calculations it follows that $\mathbf{A}$ is row equivalent to

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \dfrac{\omega_1}{\omega_2}\zeta \\ 0 & 0 & -\omega_2 \sin(\omega_2) - \omega_1 \sinh(\omega_1) & \omega_1(\cosh(\omega_1) - \zeta \cos(\omega_2)) \\ 0 & 0 & \cos(\omega_2) - \zeta \cosh(\omega_1) & \zeta\left(\sinh(\omega_1) - \dfrac{\omega_1}{\omega_2} \sin(\omega_2)\right) \end{pmatrix}. \tag{56}$$

It should be observed that, for the case $r^4 s^2 \lambda \neq 1$, a solution can only exist if the determinant of $\tilde{\mathbf{A}}$ is equal to zero. By putting this determinant equal to zero the characteristic equation is obtained, and is given by (see also Ref. [17])

$$h_{rs}(\lambda) \equiv 2 + (2 + r^4(1 - s^2)^2 \lambda) \cosh(\omega_1) \cos(\omega_2) - \left(\frac{r^2 \sqrt{\lambda}(1 + s^2)}{\sqrt{1 - r^4 s^2 \lambda}} \right) \sinh(\omega_1) \sin(\omega_2) = 0. \tag{57}$$

Now, the eigenvalues λ_n, such that $r^4 s^2 \lambda_n \neq 1$, are given implicitly by the positive roots of the characteristic equation. Now it will be shown that an eigenvalue of problems (41)–(45) can have two independent eigenfunctions. From Eq. (56) it follows that a double eigenvalue can only exist if the entries of the lower right 2×2 submatrix are equal to zero (see also Ref. [19]). Now this case will be considered. From $\tilde{\mathbf{A}}_{33} = \tilde{\mathbf{A}}_{44} = 0$ (where $\tilde{\mathbf{A}}_{ij}$ is the (i,j)-entry in $\tilde{\mathbf{A}}$) it follows that $\sinh(\omega_1) = \sin(\omega_2) = 0$. Consequently, it follows that $\cosh(\omega_1) = \pm 1$ and $\cos(\omega_2) = \pm 1$. Then from $\tilde{\mathbf{A}}_{34} = \tilde{\mathbf{A}}_{43} = 0$ it can be concluded that a double eigenvalue can only exist if $s^2 = 1$, $\omega_1 = in\pi$, where $n \in \mathbb{N}$, and $\omega_2 = m\pi$, where $m = n + 2k + 1$, and $k \in \mathbb{Z}$. Finally, from $\omega_1 = in\pi, \omega_2 = m\pi$, $s^2 = 1$, and Eq. (49), it follows that double eigenvalues can only exist if

$$2\sqrt{\lambda} = (m^2 - n^2)\pi^2, \tag{58}$$

$$2r^2 \lambda = (n^2 + m^2)\pi^2, \tag{59}$$

where $m = n + 2k - 1$ and $n, k \in \mathbb{N}$. Hence it follows that $\lambda = \pi^4/4(m^2 - n^2)^2$, where $m = n + 2k - 1$ and $n, k \in \mathbb{N}$, is an double eigenvalue if $r^2 = 2(n^2 + m^2)/\pi^2(m^2 - n^2)^2$ and $s^2 = 1$.

Now the eigenfunctions for the case $r^4 s^2 \lambda \neq 1$ will be considered. The eigenfunctions corresponding to simple eigenvalues $\lambda_n \neq 1/r^4 s^2$ have been given in Ref. [17], and are given by $\hat{\mathbf{\Phi}}_n(x) = [U_n(x), \Psi_n(x)]^{\mathrm{T}}$, where

$$U_n(x) = D_n\left[\left(\cosh(\omega_{1,n}) - \frac{1}{\zeta_n}\cos(\omega_{2,n})\right)(\cosh(\omega_{1,n}x) - \cos(\omega_{2,n}x)) - \left(\frac{\omega_{2,n}}{\omega_{1,n}}\sinh(\omega_{1,n}) - \sin(\omega_{2,n})\right)\left(\frac{\omega_{1,n}}{\omega_{2,n}}\sinh(\omega_{1,n}x) + \frac{1}{\zeta_n}\sin(\omega_{2,n}x)\right)\right], \quad (60)$$

$$\Psi_n(x) = H_n\left[\left(\frac{1}{\zeta_n}\cosh(\omega_{1,n}) - \cos(\omega_{2,n})\right)(\cosh(\omega_{1,n}x) - \cos(\omega_{2,n}x)) - \left(\frac{\omega_{1,n}}{\omega_{2,n}}\sinh(\omega_{1,n}) + \sin(\omega_{2,n})\right)\left(\frac{\omega_{2,n}}{\omega_{1,n}}\frac{1}{\zeta_n}\sinh(\omega_{1,n}x) - \sin(\omega_{2,n}x)\right)\right], \quad (61)$$

where ζ_n is given by Eq. (55), and where D_n and H_n are connected by Eqs. (47)–(48) and (50)–(53). The general solution of Eqs. (41)–(45) corresponding to a double eigenvalue $\lambda_n \neq 1/r^4 s^2$ is given by $\hat{\mathbf{\Phi}}_n(x) = [U_n(x), \Psi_n(x)]^{\mathrm{T}}$, where

$$U_n(x) = D_{1,n}\left(\frac{\omega_{1,n}}{\omega_{1,n}^2 + r^2 s^2 \lambda_n}\sinh(\omega_{1,n}x) + \frac{\omega_{2,n}}{r^2 s^2 \lambda_n - \omega_{2,n}^2}\sin(\omega_{2,n}x)\right) + D_{2,n}\left(\frac{-\omega_1}{\omega_{1,n}^2 + r^2 s^2 \lambda_n}\cosh(\omega_{1,n}x) + \frac{\zeta\omega_{1,n}}{r^2 s^2 \lambda_n - \omega_{2,n}^2}\cos(\omega_{2,n}x)\right), \quad (62)$$

$$\Psi_n(x) = H_{1,n}(\cos(\omega_{2,n}x) - \cosh(\omega_{1,n}x)) + H_{2,n}\left(\sinh(\omega_{1,n}x) - \zeta\frac{\omega_{1,n}}{\omega_{2,n}}\sin(\omega_{2,n}x)\right), \quad (63)$$

where $D_{1,n}, D_{2,n}, H_{1,n}$ and $H_{2,n}$ are connected by Eqs. (47), (48) and (50)–(53). Now, by putting $D_{1,n} = 1$ and $D_{2,n} = 0$ into Eqs. (62) and (63), and by putting $D_{1,n} = 0$ and $D_{2,n} = 1$ into Eqs. (62) and (63) two independent eigenfunction are found. Note that the values of $H_{1,n}$ and $H_{2,n}$ follow immediately from the values of $D_{1,n}$ and $D_{2,n}$, and Eqs. (50)–(53). These independent eigenfunctions are not necessarily orthogonal. But two independent eigenfunctions corresponding to a double eigenvalue can be chosen orthogonal. The Gram–Schmidt orthogonalization method can be used to accomplish this.

Now the case $r^4 s^2 \lambda = 1$ (i.e. $\omega_1 = 0$) will be considered. Substitute $r^4 s^2 \lambda = 1$ into Eqs. (41) and (42) to obtain

$$r^2 s^2 \Psi'' + U' = 0, \quad (64)$$

$$r^2(U'' - \Psi') = -U. \quad (65)$$

The solution of Eqs. (64) and (65) is given by $\hat{\mathbf{\Phi}}(x) = [U(x), \Psi(x)]^{\mathrm{T}}$, where

$$U(x) = c_0 + c_2\cos(\mu x) + c_3\sin(\mu x), \quad (66)$$

$$\Psi(x) = d_0 + d_1 x + d_2\cos(\mu x) + d_3\sin(\mu x), \quad (67)$$

where $\mu = \sqrt{1 + s^2}/rs$, and where the constants c_0, c_2, c_3, and d_i, where $i = 0, 1, 2, 3$, are unknown so far. From Eq. (65) it follows that $c_0 = r^2 d_1$, $c_2 = -r^2 s^2 \mu d_3$, and $c_3 = r^2 s^2 \mu d_2$. Then, by elementary calculations, it follows that a solution of Eqs. (64), (65) and (43)–(45) can only exist if $\hat{\mathbf{A}}\mathbf{d} = \mathbf{0}$, where $\mathbf{d} = [d_0, d_3, d_2, d_1]^{\mathrm{T}}$, and where

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \dfrac{-1}{s^2\mu} \\ 0 & 0 & -\mu\sin(\mu) & 1+\dfrac{\cos(\mu)}{s^2} \\ 0 & 0 & s^2\cos(\mu)+1 & -1+\dfrac{\sin(\mu)}{\mu} \end{pmatrix}. \tag{68}$$

Now, it will be studied for which values of s and r double eigenvalues can occur. Double eigenvalues can only exist if the entries of the lower right 2×2 submatrix of $\hat{\mathbf{A}}$ are equal to zero. Since $\mu = \sqrt{1+s^2}/rs > 0$ it follows from Eq. (68) that $\hat{\mathbf{A}}_{44} < 0$. Hence double eigenvalues are not possible for the case $r^4 s^2 \lambda = 1$. Then, from Eq. (68), it can be concluded that a solution of problem Eqs. (64), (65) and (43)–(45) can only exist if r and s satisfy the following characteristic equation:

$$2s^2 + (1+s^4)\cos(\mu) = s^2\mu\sin(\mu), \tag{69}$$

where $\mu = \sqrt{1+s^2}/rs$. Note that this equation also follows from Eq. (57) by taking the limit $r^4s^2\lambda \to 1$. Therefore, $\lambda = 1/r^4s^2$ can only be an eigenvalue of problems (41)–(45) if it satisfies Eq. (57). Hence all the eigenvalues of problems (41)–(45) are given by the roots of Eq. (57). The eigenfunction corresponding to the eigenvalue $\lambda = 1/r^4s^2$ is given by $\hat{\mathbf{\Phi}}(x) = [U(x), \Psi(x)]^{\mathrm{T}}$, where

$$U(x) = D[\sin(\mu)(1-\cos(\mu x)) + (s^2+\cos(\mu))\sin(\mu x)], \tag{70}$$

$$\Psi(x) = H\left[(s^2+\cos(\mu))(\cos(\mu x)-1) + \mu\sin(\mu)\left(\frac{\sin(\mu x)}{\mu} + s^2 x\right)\right], \tag{71}$$

and where D and H are connected by $c_0 = r^2 d_1$, $c_2 = -r^2 s^2 \mu d_3$, $c_3 = r^2 s^2 \mu d_2$, Eqs. (66) and (67).

So far, it has been found that the eigenvalues of problems (41)–(45) are given implicitly by the positive roots of Eq. (57). In Ref. [20] it has been shown that problems (41)–(45) has infinitely many, isolated eigenvalues which all have a finite multiplicity. Now the nth positive eigenvalue (counting multiplicities) of problems (41)–(45) will be denoted by λ_n. Furthermore, for each simple eigenvalue an eigenfunction $\hat{\mathbf{\Phi}}_n(x) = [U_n(x), \Psi_n(x)]^{\mathrm{T}}$ has been found, which is given by Eqs. (60) and (61) for the case $r^4s^2\lambda_n \neq 1$, and by Eqs. (70) and (71) for the case $r^4s^2\lambda_n = 1$. In addition, it has been argued that for each double eigenvalue λ_n two orthogonal eigenfunction can be obtained from Eqs. (62) and (63). In the previous section it has been shown that the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the inner product defined by Eq. (33). Hence the eigenfunctions $\hat{\mathbf{\Phi}}_n(x) = [U_n(x), \Psi_n(x)]^{\mathrm{T}}$ corresponding to the eigenvalues λ_n of problems (41)–(45) form an orthogonal set with respect to the inner product defined by (33). Now, the solution of initial boundary value problems (15)–(21) with $\varepsilon = 0$ will be constructed. From $T_n'' + \lambda_n T_n = 0$ the function $T_n(t)$ can be determined for each eigenvalue λ_n. So, infinitely many non-trivial solutions of problems (15)–(19) with $\varepsilon = 0$ have been determined. Using the superposition principle and the initial values (20) and (21) the solution $\mathbf{\Gamma}(x,t) = [u(x,t), \psi(x,t)]^{\mathrm{T}}$ of the initial-boundary value problems (15)–(21) with $\varepsilon = 0$ is obtained, and is given by

$$\mathbf{\Gamma}(x,t) = \sum_{n=1}^{\infty} (A_n\cos(\sqrt{\lambda_n}t) + B_n\sin(\sqrt{\lambda_n}t))\mathbf{\Phi}_n(x) = \sum_{n=1}^{\infty} T_n(t)\mathbf{\Phi}_n(x), \tag{72}$$

where λ_n is the nth positive root (counting multiplicities) of the characteristic equation (57), and where

$$A_n = \int_0^1 f(x)\phi_n(x) + r^2 p(x)\varphi_n(x)\,\mathrm{d}x, \tag{73}$$

$$B_n = \frac{1}{\sqrt{\lambda_n}}\int_0^1 h(x)\phi_n(x) + r^2 q(x)\varphi_n(x)\,\mathrm{d}x, \tag{74}$$

and where $\mathbf{\Phi}_n(x) = [\phi_n(x), \psi_n(x)]^{\mathrm{T}}$ is given by

$$\mathbf{\Phi}_n(x) = \frac{\hat{\mathbf{\Phi}}_n(x)}{\left(\int_0^1 \{U_n^2 + r^2 \Psi_n^2\}\,\mathrm{d}x\right)^{1/2}}, \tag{75}$$

where $\hat{\mathbf{\Phi}}_n(x)$, for the case that λ_n is a simple eigenvalue, is given by Eqs. (60) and (61) for the case $r^4 s^2 \lambda_n \neq 1$, and by Eqs. (70) and (71) for the case $r^4 s^2 \lambda_n = 1$, and where $\hat{\mathbf{\Phi}}_n(x)$, for the case that λ_n is a double eigenvalue can be obtained from Eqs. (62) and (63). The eigenfunctions $\mathbf{\Phi}_n(x)$ form an orthonormal set with respect to the inner product defined by Eq. (33).

Now approximations of the roots of the characteristic equation (57) will be constructed. These approximate forms of the roots λ_n will be used in Section 5 to determine the effect of the self-weight of the beam on the natural frequencies. It should be observed that the case $r^4 s^2 \lambda \ll 1$ and $r^4(1-s^2)^2\lambda \ll 1$, and the case $r^4 s^2 \lambda \gg 1$ and $r^4(1-s^2)^2\lambda \gg 1$ can be distinguished. In this paper, for simplicity, only the influence of r and λ on the approximate forms will be studied. It will be assumed that $s^2 = E/k'G$ (the ratio of the Young modulus to the shear coefficient depending on the shape of the cross-section multiplied by the modulus of elasticity in shear) is fixed and not equal to one. Hence the case $r^4\lambda \ll 1$ and the case $r^4\lambda \gg 1$ can be studied instead of the case $r^4 s^2 \lambda \ll 1$ and $r^4(1-s^2)^2\lambda \ll 1$, and the case $r^4 s^2\lambda \gg 1$ and $r^4(1-s^2)^2\lambda \gg 1$, respectively. Firstly the case $r^4\lambda \ll 1$ will be considered. For this case it follows by straightforward calculations that Definition (57) is approximately given by

$$h_{rs}(\lambda) = 2(1 + \cosh(\sqrt[4]{\lambda})\cos(\sqrt[4]{\lambda})) + (1 + \sqrt[4]{\lambda})\mathcal{O}(r^2\sqrt{\lambda}e^{\sqrt[4]{\lambda}}), \tag{76}$$

and Eq. (49) by $\omega_{1,2} = \sqrt[4]{\lambda}(1 + \mathcal{O}(r^2\sqrt{\lambda}))$. Eq. (57) with $r = 0$ is exactly the characteristic equation of the cantilevered Euler–Bernoulli beam. The roots of Eq. (57) with $r = 0$ are given by $\sqrt[4]{\lambda_1} = 1.8751$, $\sqrt[4]{\lambda_2} = 4.6941$, and for the higher values $\sqrt[4]{\lambda_n} \approx (n - 1/2)\pi$. Now the case $r^4\lambda \gg 1$ will be discussed. For this case it follows by straightforward calculations that Definition (57) is approximately given by

$$h_{rs}(\lambda) = r^4(1-s^2)^2\lambda\left(\cosh(\omega_1)\cos(\omega_2) + \mathcal{O}\left(\frac{1}{r^4\lambda}\right)\right), \tag{77}$$

and that $\omega_1^2 = r^2\lambda(-s^2 + (1/r^4(1-s^2)\lambda) + \mathcal{O}(1/r^8\lambda^2))$, and $\omega_2^2 = r^2\lambda(1 + (1/r^4(1-s^2)\lambda) + \mathcal{O}(1/r^8\lambda^2))$. Now approximations of the eigenvalues λ_n will be constructed. From Eq. (77) it follows that the case $h_{1_{rs}}(\lambda) \equiv \cosh(\omega_1) + \mathcal{O}(1/r^4\lambda) = 0$, and the case $h_{2_{rs}}(\lambda) \equiv \cos(\omega_2) + \mathcal{O}(1/r^4\lambda) = 0$ have to be considered. Note that the case that $\cosh(\omega_1)$ and $\cos(\omega_2)$ are both close to zero should also be considered. Since for this case it is much more difficult to find asymptotic approximations of the eigenvalues this case will not be studied any further in this paper. From $h_{1_{rs}}(\lambda) = 0$ it follows that $i\omega_{1,n} = \hat{\omega}_{1,n} = (n - \frac{1}{2})\pi + \mathcal{O}(1/r^4\lambda)$. And from $h_{2_{rs}}(\lambda) = 0$ it follows that $\omega_{2,n} = (n - \frac{1}{2})\pi + \mathcal{O}(1/r^4\lambda_n)$. The eigenvalues λ_n corresponding to $h_{1_{rs}}(\lambda) = 0$ ($h_{2_{rs}}(\lambda) = 0$) will be denoted by $\lambda_{1,n}$ ($\lambda_{2,n}$). Now it follows that $\sqrt{\lambda_{1,n}} = (n - \frac{1}{2})\pi/rs(1 + \mathcal{O}(1/r^2n^2))$ and $\sqrt{\lambda_{2,n}} = (n - \frac{1}{2})\pi/r(1 + \mathcal{O}(1/r^2n^2))$. For similar estimates see also Ref. [20].

5. Formal approximations

In this section the vibrations of a standing, uniform Timoshenko beam which is clamped at one end and free at the other end will be considered. An approximation of the solution of the initial-boundary value problems (15)–(21) will be constructed by using a two time-scales perturbation method. Conditions like $t > 0, t \geq 0, 0 < x < 1$ will be dropped for abbreviation. By expanding the unknown functions $u(x,t)$ and $\psi(x,t)$ in a Taylor series with respect to ε it follows that

$$u(x,t;\varepsilon) = \hat{u}_0(x,t) + \varepsilon\hat{u}_1(x,t) + \varepsilon^2\hat{u}_2(x,t) + \cdots, \tag{78}$$

$$\psi(x,t;\varepsilon) = \hat{\psi}_0(x,t) + \varepsilon\hat{\psi}_1(x,t) + \varepsilon^2\hat{\psi}_2(x,t) + \cdots. \tag{79}$$

It is assumed that the functions $\hat{u}_i(x,t)$ and $\hat{\psi}_i(x,t)$ are $\mathcal{O}(1)$ on time-scales of order $1/\varepsilon$. The approximation of the solution will contain secular terms. Since $\hat{u}_i(x,t)$ and $\hat{\psi}_i(x,t)$ are assumed to be $\mathcal{O}(1)$, and because the solutions are bounded, secular terms should be avoided when approximations are constructed on a time-scale of $\mathcal{O}(\varepsilon^{-1})$. That is why a two time-scales perturbation method is applied. Using such a two time-scales perturbation method the functions $u(x,t)$ and $\psi(x,t)$ are supposed to be a function of x, t, and $\tau = \varepsilon t$. So put

$$u(x,t) = w(x,t,\tau;\varepsilon), \tag{80}$$

$$\psi(x,t) = \varphi(x,t,\tau;\varepsilon). \tag{81}$$

A result of this is that

$$u_t = w_t + \varepsilon w_\tau, \tag{82}$$

$$u_{tt} = w_{tt} + 2\varepsilon w_{t\tau} + \varepsilon^2 w_{\tau\tau}, \tag{83}$$

$$\psi_t = \varphi_t + \varepsilon\varphi_\tau, \tag{84}$$

$$\psi_{tt} = \varphi_{tt} + 2\varepsilon\varphi_{t\tau} + \varepsilon^2\varphi_{\tau\tau}. \tag{85}$$

Substitution of Eqs. (80)–(85) into problems (15)–(21) yields

$$\varphi_{xx} + \left(\frac{1}{r^2 s^2}\right)(w_x - \varphi) - r^2\varphi_{tt} = 2r^2\varepsilon\varphi_{t\tau} + r^2\varepsilon^2\varphi_{\tau\tau}, \tag{86}$$

$$\left(\frac{1}{r^2 s^2}\right)(w_{xx} - \varphi_x) - w_{tt} = 2\varepsilon w_{t\tau} + \varepsilon^2 w_{\tau\tau} + \varepsilon[(1-x)w_x]_x, \tag{87}$$

$$w(0,t,\tau;\varepsilon) = \varphi(0,t,\tau;\varepsilon) = 0, \tag{88}$$

$$\varphi_x(1,t,\tau;\varepsilon) = 0, \tag{89}$$

$$w_x(1,t,\tau;\varepsilon) - \varphi(1,t,\tau;\varepsilon) = 0, \tag{90}$$

$$w(x,0,0;\varepsilon) = f(x) \quad \text{and} \quad w_t(x,0,0;\varepsilon) = h(x) - \varepsilon w_\tau(x,0,0;\varepsilon), \tag{91}$$

$$\varphi(x,0,0;\varepsilon) = p(x) \quad \text{and} \quad \varphi_t(x,0,0;\varepsilon) = q(x) - \varepsilon\varphi_\tau(x,0,0;\varepsilon). \tag{92}$$

Assuming that

$$w(x,t,\tau;\varepsilon) = u_0(x,t,\tau) + \varepsilon u_1(x,t,\tau) + \varepsilon^2 u_2(x,t,\tau) + \cdots, \tag{93}$$

$$\varphi(x,t,\tau;\varepsilon) = \psi_0(x,t,\tau) + \varepsilon\psi_1(x,t,\tau) + \varepsilon^2\psi_2(x,t,\tau) + \cdots, \tag{94}$$

then by collecting terms of equal powers in ε it follows from Eqs. (86)–(92) that the $\mathcal{O}(1)$-problem is:

$$\psi_{0_{xx}} + \left(\frac{1}{r^2 s^2}\right)(u_{0_x} - \psi_0) - r^2\psi_{0_{tt}} = 0, \tag{95}$$

$$\left(\frac{1}{r^2 s^2}\right)(u_{0_{xx}} - \psi_{0_x}) - u_{0_{tt}} = 0, \tag{96}$$

$$u_0(0,t) = \psi_0(0,t) = 0, \tag{97}$$

$$\psi_{0_x}(1,t) = 0, \tag{98}$$

$$u_{0_x}(1,t) - \psi_0(1,t) = 0, \tag{99}$$

$$u_0(x,0,0) = f(x) \quad \text{and} \quad u_{0_t}(x,0,0) = h(x), \tag{100}$$

$$\psi(x,0,0) = p(x) \quad \text{and} \quad \psi_{0_t}(x,0,0) = q(x) \tag{101}$$

and that the $\mathcal{O}(\varepsilon)$-problem is:

$$\psi_{1_{xx}} + \left(\frac{1}{r^2 s^2}\right)(u_{1_x} - \psi_1) - r^2 \psi_{1_{tt}} = 2r^2 \psi_{0_{t\tau}}, \tag{102}$$

$$\left(\frac{1}{r^2 s^2}\right)(u_{1_{xx}} - \psi_{1_x}) - u_{1_{tt}} = 2u_{0t\tau} + [(1-x)u_{0_x}]_x, \tag{103}$$

$$u_1(0,t) = \psi_1(0,t) = 0, \tag{104}$$

$$\psi_{1_x}(1,t) = 0, \tag{105}$$

$$u_{1_x}(1,t) - \psi_1(1,t) = 0, \tag{106}$$

$$u_1(x,0,0) = 0 \quad \text{and} \quad u_{1_t}(x,0,0) = -u_{0_\tau}(x,0,0), \tag{107}$$

$$\psi_1(x,0,0) = 0 \quad \text{and} \quad \psi_{1_t}(x,0,0) = -\psi_{0_\tau}(x,0,0). \tag{108}$$

The solution $\mathbf{\Gamma}_0(x,t,\tau) = [u_0(x,t,\tau), \psi_0(x,t,\tau)]^{\mathrm{T}}$ of the $\mathcal{O}(1)$-problems (95)–(101) has been determined in the previous section and is given by

$$\mathbf{\Gamma}_0(x,t,\tau) = \sum_{n=1}^{\infty} T_{0n}(t,\tau)\mathbf{\Phi}_n(x), \tag{109}$$

where $T_{0n}(t,\tau) = A_{0n}(\tau)\cos(\sqrt{\lambda_n}t) + B_{0n}(\tau)\sin(\sqrt{\lambda_n}t)$, where $\mathbf{\Phi}_n(x) = [\phi_n(x), \varphi_n(x)]^{\mathrm{T}}$ is given by Eq. (75), and where

$$A_{0n}(0) = \int_0^1 f(x)\phi_n(x) + r^2 p(x)\varphi_n(x)\,\mathrm{d}x, \tag{110}$$

$$B_{0n}(0) = \frac{1}{\sqrt{\lambda_n}} \int_0^1 h(x)\phi_n(x) + r^2 q(x)\varphi_n(x)\,\mathrm{d}x. \tag{111}$$

Since the unknown function $\mathbf{\Gamma}_1(x,t) = [u_1(x,t,\tau), \psi_1(x,t,\tau)]^{\mathrm{T}}$ satisfies the same boundary conditions as $\mathbf{\Gamma}_0(x,t,\tau)$, it is assumed that the solution of problems (102)–(108) is given by

$$\mathbf{\Gamma}_1(x,t,\tau) = \sum_{n=1}^{\infty} T_{1n}(t,\tau)\mathbf{\Phi}_n(x), \tag{112}$$

where $\mathbf{\Phi}_n(x) = [\phi_n(x), \varphi_n(x)]^{\mathrm{T}}$ is given by Eq. (75). Now an equation for the unknown function $T_{1n}(t,\tau)$ will be determined in the following way: Firstly, substitute Eq. (112) into Eqs. (102) and (103) and multiply the so-obtained equations by $\varphi_n(x)$ and $\phi_n(x)$, respectively. Then sum the so-obtained equations. Finally, integrate the so-obtained equation with respect to x form 0 to 1, and use the orthogonality of the eigenfunctions $\mathbf{\Phi}_n(x) = [\phi_n(x), \varphi_n(x)]^{\mathrm{T}}$, to obtain:

$$T_{1n_{tt}}(t,\tau) + \lambda_n T_{1n}(t,\tau) = -2T_{0n_{t\tau}}(t,\tau) + \sum_{m=1}^{\infty} \Theta_{mn} T_{0m}(t,\tau), \tag{113}$$

where

$$\Theta_{mn} = \int_0^1 (1-x)\phi_{m_x}(x)\phi_{n_x}(x)\,\mathrm{d}x, \tag{114}$$

and where $T_{0n}(t,\tau) = A_{0n}(\tau)\cos(\sqrt{\lambda_n}t) + B_{0n}(\tau)\sin(\sqrt{\lambda_n}t)$. From $T_{0n}(t,\tau)$ it follows that $T_{0n}(t,\tau)$ and $T_{0n_{t\tau}}(t,\tau)$ are solutions of the homogeneous equation corresponding to Eq. (113), and that $T_{0m}(t,\tau)$ with $m \neq n$ are not solutions of the homogeneous equation corresponding to Eq. (113). Therefore, the right-hand side of Eq. (113) contains terms which are solutions of the homogeneous equation corresponding to Eq. (113). These terms will give rise to unbounded terms, the so-called secular terms, in the solution $T_{1n}(t,\tau)$ of Eq. (113). Since it is assumed in the asymptotic expansions that the functions $u_0(x,t,\tau)$, $\psi_0(x,t,\tau)$, $u_1(x,t,\tau)$, $\psi_1(x,t,\tau)$, $u_2(x,t,\tau)$, $\psi_2(x,t,\tau),\ldots$ are bounded on time-scales of $\mathcal{O}(\varepsilon^{-1})$ these secular terms should be avoided. In $T_{0n}(t,\tau)$ the functions $A_{0n}(\tau)$ and $B_{0n}(\tau)$ are still undetermined. These functions will be used to avoid secular terms in the solution of Eq. (113) in the following way. Let the sum of the terms in the right-hand side of Eq. (113) that give rise to secular terms in the solution of Eq. (113)be equal to zero, yielding

$$-2T_{0n_{t\tau}}(t,\tau)+\Theta_{nn}T_{0n}(t,\tau)=0. \tag{115}$$

By substituting $T_{0n}(t,\tau)$ into Eq. (115) the following system of coupled differential equations for the functions $A_{0n}(\tau)$ and $B_{0n}(\tau)$ can be obtained:

$$A_{0n_\tau}(\tau)=-\frac{\Theta_{nn}}{2\sqrt{\lambda_n}}B_{0n}(\tau), \tag{116}$$

$$B_{0n_\tau}(\tau)=\frac{\Theta_{nn}}{2\sqrt{\lambda_n}}A_{0n}(\tau), \tag{117}$$

where $A_{0n}(0)$ and $B_{0n}(0)$ are given by Eqs. (110) and (111), respectively. From Eqs. (116) and (117) the functions $A_{0n}(\tau)$ and $B_{0n}(\tau)$ can be determined and are given by

$$A_{0n}(\tau)=A_{0n}(0)\cos\left(\frac{\Theta_{nn}\tau}{2\sqrt{\lambda_n}}\right)-B_{0n}(0)\sin\left(\frac{\Theta_{nn}\tau}{2\sqrt{\lambda_n}}\right), \tag{118}$$

$$B_{0n}(\tau)=B_{0n}(0)\cos\left(\frac{\Theta_{nn}\tau}{2\sqrt{\lambda_n}}\right)+A_{0n}(0)\sin\left(\frac{\Theta_{nn}\tau}{2\sqrt{\lambda_n}}\right), \tag{119}$$

respectively. By substituting $A_{0n}(\tau)$ and $B_{0n}(\tau)$ into $T_{0n}(t,\tau)$ it follows that

$$T_{0n}(t,\tau)=A_{0n}(0)\cos\left(\sqrt{\lambda_n}t-\frac{\Theta_{nn}\tau}{2\sqrt{\lambda_n}}\right)+B_{0n}(0)\sin\left(\sqrt{\lambda_n}t-\frac{\Theta_{nn}\tau}{2\sqrt{\lambda_n}}\right), \tag{120}$$

where Θ_{nn} is given by Eq. (114). Now, an $\mathcal{O}(\varepsilon)$-approximation of the solution of the initial-boundary value problems (15)–(21) has been determined. This $\mathcal{O}(\varepsilon)$-approximation is given by Eq. (109), and is valid on time-scales of $\mathcal{O}(\varepsilon^{-1})$. It is beyond the scope of this paper to prove that the $\mathcal{O}(\varepsilon)$-approximation are indeed valid on time-scales of $\mathcal{O}(\varepsilon^{-1})$.

From Eq. (120) it follows that an approximation of the frequency ($\omega_n(\varepsilon)$) of the nth mode of a standing Timoshenko beam in a gravity-field is given by

$$\omega_n(\varepsilon)=\sqrt{\lambda_n}-\frac{\varepsilon\Theta_{nn}}{2\sqrt{\lambda_n}}, \tag{121}$$

where $\varepsilon=g\rho AL^3/EI$, Θ_{nn} is given by Eq. (114), and $\sqrt{\lambda_n}$ is the frequency of the nth mode of a gravity-free Timoshenko beam, which is given by the squareroot of the nth positive root of Eq. (57). Note that the order of the highest derivatives (with respect to x and t) that appears in problems (15)–(21) with $\varepsilon=0$ and problems (15)–(21) with $\varepsilon\neq 0$ are the same. Therefore, it is assumed that $\omega_n(\varepsilon)$ is an $\mathcal{O}(\sqrt{\lambda_n}\varepsilon^2)$-approximation of the magnitude of the frequency. Due to gravity and the self-weight of the beam a linearly varying compression force is acting on the beam. The second term of the right-hand side of Eq. (121) (i.e. $\varepsilon\Theta_{nn}/2\sqrt{\lambda_n}$) represents the influence of this compression on the frequency of the nth mode of the beam. Since $\Theta_{nn}>0$ it follows from Eq. (121) that the inclusion of the compression force in the beam model reduces the magnitude of the frequency. Now we will study this decrease in magnitude of the frequency. Note that the value of $\varepsilon\Theta_{nn}/2\sqrt{\lambda_n}$ depends on the parameters ε, r, and s and the mode number n. Firstly, it should be observed that the frequency ($\omega_n(\varepsilon)$) reduces by increasing values of ε. Now the influence of r,s, and n on the decrease in magnitude of the frequencies will be discussed. In the previous section approximate forms the eigenvalues λ_n have been constructed for the case $r^4\lambda_n\ll 1$, and for the case $r^4\lambda_n\gg 1$. Therefore, the values of the frequencies will be considered for the case $r^4\lambda_n\ll 1$, the case $r^4\lambda_n\approx 1$, and the case $r^4\lambda_n\gg 1$. Firstly, the case $r^4\lambda_n\ll 1$ will be studied. Now the characteristic equation (57) can be approximated by Eq. (57) with $r=0$. This is the characteristic equation of a cantilevered Euler–Bernoulli beam. The integrand in Θ_{nn} is given by $(1-x)\phi_n^2(x)$, where $\phi_n(x)$ is given by Eq. (75). Now, $\phi_n(x)$ can be approximated by the nth eigenfunction

Table 1
Numerical approximations of $\sqrt{\lambda_n}$ and $\Theta_{nn}/2\sqrt{\lambda_n}$ for the case $r^2 = 0.01$ and $s^2 = 2.8$ and for the case $r^2 = 0.001$ and $s^2 = 0.5$

n	$r^2 = 0.01, s^2 = 2.8$		$r^2 = 0.001$, $s^2 = 0.5$	
	$\sqrt{\lambda_n}$	$\left(\frac{\Theta_{nn}}{2\sqrt{\lambda_n}}\right)$	$\sqrt{\lambda_n}$	$\left(\frac{\Theta_{nn}}{2\sqrt{\lambda_n}}\right)$
1	3.2471	0.2263	3.5038	0.22303
2	14.803	0.2468	21.519	0.19390
3	32.415	0.3377	58.448	0.19687
4	49.649	0.3934	109.98	0.20532
5	65.263	0.2922	173.47	0.07178
6	70.555	0.2604	246.27	0.01801
7	84.075	0.3349	326.22	0.00193
8	92.021	0.4562	411.60	0.00292
9	105.87	0.3324	501.12	0.00409
10	113.75	0.6524	593.77	0.00754

corresponding to the cantilevered Euler–Bernoulli beam. By using this eigenfunction the integral Θ_{nn} can be approximated by (see Refs. [2,6])

$$\frac{\Theta_{nn}}{2\sqrt{\lambda_n}} = \frac{1}{4\sqrt{\lambda_n}}((1 + \sqrt[4]{\lambda_n}\chi_n)^2 + 3), \tag{122}$$

where $\chi_n = (\sin(\sqrt[4]{\lambda_n}) - \sinh(\sqrt[4]{\lambda_n}))/(\cos(\sqrt[4]{\lambda_n}) + \cosh(\sqrt[4]{\lambda_n}))$. It should be observed that the value of $\Theta_{nn}/2\sqrt{\lambda_n}$ becomes small compared to the value of $\sqrt{\lambda_n}$ for increasing values of the mode number n. Hence it can be concluded that the decrease in magnitude of the frequency (due to the compression force) will become relatively small (compared to $\omega_n(\varepsilon)$) by increasing mode number n. Furthermore, from Eqs. (121) and (122) it follows that the parameters r^2 and s^2 do not significantly change the frequencies of the oscillation modes when $r^4\lambda_n \ll 1$.

For the case $r^4\lambda_n \approx 1$ numerical methods can be used to determine the value of $\Theta_{nn}/2\sqrt{\lambda_n}$. In Table 1 the first ten values of $\Theta_{nn}/2\sqrt{\lambda_n}$ are listed for the case $r^2 = 0.01$ and $s^2 = 2.8$, and for the case $r^2 = 0.001$ and $s^2 = 0.5$. For the modes listed in Table 1 the decrease in magnitude of the frequencies due to the compression force becomes relatively small (compared to $\omega_n(\varepsilon)$) by increasing mode number.

Now consider the case $r^4\lambda_n \gg 1$, that is, consider the higher order modes. It should be observed that the eigenfunctions $\mathbf{\Phi}_n(x) = [\phi_n(x), \psi_n(x)]^{\mathrm{T}}$ for this case are given by Eq. (75), where $U_n(x)$ and $\Psi_n(x)$ are given by Eqs. (60) and (61), respectively. In Section 4 it has been observed that for $r^4\lambda_n \gg 1$ two sets of roots of the characteristic equation (57) can be distinguished. The roots of the first (second) set are denoted by $\lambda_{1,n}$ ($\lambda_{2,n}$). Now the value of the approximation of the frequencies ($\omega_n(\varepsilon)$), will be studied for these two sets. The roots of the first set are given by $\sqrt{\lambda_{1,n}} = [(n - \frac{1}{2})\pi/rs] + \mathcal{O}(1/r^3 n)$ (see Section 4). Now, it can be shown, by elementary calculations, that $\Theta_{nn} = \hat{\omega}^2_{1,n}/2(1 + \mathcal{O}(1/r^3\sqrt{\lambda_{1,n}}))$. Consequently, from $\hat{\omega}^2_{1,n} = r^2\lambda_{1,n}(s^2 + \mathcal{O}(1/r^4\lambda_{1,n}))$, it follows that

$$\frac{\Theta_{nn}}{2\sqrt{\lambda_{1,n}}} = \frac{rs\left(n - \frac{1}{2}\right)\pi}{4}\left(1 + \mathcal{O}\left(\frac{1}{r^2 n}\right)\right). \tag{123}$$

From Eqs. (121) and (123) it follows that the decrease in magnitude of the frequency due to the compression force increases by increasing mode number n. Note that this is not the case for a vertical, cantilevered Euler–Bernoulli beam (see Refs. [2,6]). Hence the inclusion of shear deformation and rotatory inertia increases the decrease in magnitude (due to the compression force) of the frequencies of a vertical, cantilevered beam. Now by substituting Eq. (123) and $\sqrt{\lambda_{1,n}} = (n - \frac{1}{2})\pi/rs + \mathcal{O}(1/r^3 n)$ into Eq. (121) it follows that $\omega_n(\varepsilon)$ is approximately given by

$$\omega_n(\varepsilon) = \left(\frac{\left(n - \frac{1}{2}\right)\pi}{rs}\right)\left(1 - \frac{\varepsilon r^2 s^2}{4}\right). \tag{124}$$

Thus the frequencies reduces by increasing values of the parameters ε, r, and s.

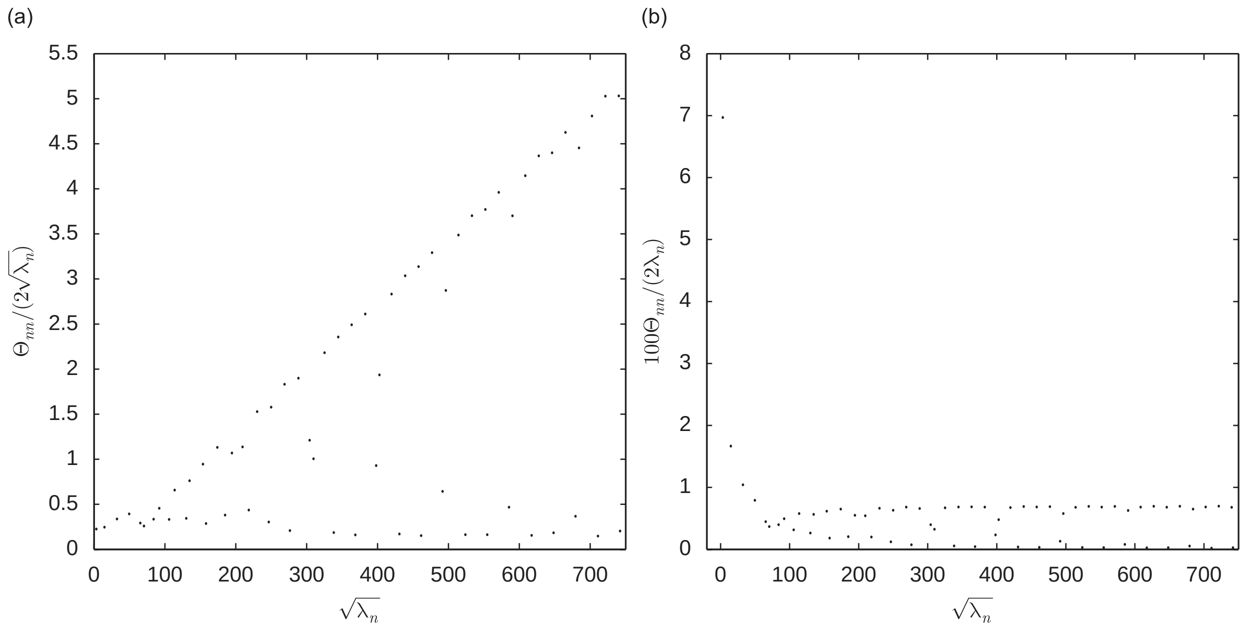

Fig. 3. (a) The effect of gravity $\Theta_{nn}/2\sqrt{\lambda_n}$ plotted against the frequency $\sqrt{\lambda_n}$ for the case $r^2 = 0.01$ and $s^2 = 2.8$. (b) The relative (compared to $\sqrt{\lambda_n}$) effect of gravity (in per cent) for the case $r^2 = 0.01$ and $s^2 = 2.8$.

For the second set of roots ($\sqrt{\lambda_{2,n}}$) of Eq. (57) it can be shown that

$$\Theta_{nn} = \left(\frac{s^2}{2r^2(1-s^2)^2}\right)\left(\frac{(1+(1/s^2))\cos^2(\hat{\omega}_{1,n}) + (\sin(\hat{\omega}_{1,n}) - s\sin(\omega_{2,n}))^2}{\cos^2(\hat{\omega}_{1,n})} + \mathcal{O}\left(\frac{1}{r^3\sqrt{\lambda_{2,n}}}\right)\right). \quad (125)$$

Eq. (125) leads to $\Theta_{nn} = \mathcal{O}(r^{-2})$. Now, since $\sqrt{\lambda_{2,n}} = ((n-\frac{1}{2})\pi/r) + \mathcal{O}(1/r^3 n)$ (see Section 4), it follows from Eq. (121) that the frequencies are approximately given by $\sqrt{\lambda_{2,n}}(1+\mathcal{O}(\varepsilon/n^2))$. Hence it follows that the decrease in the magnitude of the frequency (due to the compression force) decreases by increasing mode number n. Moreover, it can be concluded that the decrease in magnitude of the frequencies for the second set is significantly smaller compared to the decrease in magnitude of the frequencies corresponding to the first set of roots of the characteristic equation (57).

In Fig. 3(a) the values of $\Theta_{nn}/2\sqrt{\lambda_n}$ and $\sqrt{\lambda_n}$ are given for the case $r^2 = 0.01$ and $s^2 = 2.8$. From this figure it can also be observed that the for the higher order modes two sets of frequencies can be distinguished. For the first set there is a predominantly linear relationship between the values of $\Theta_{nn}/2\sqrt{\lambda_n}$ and $\sqrt{\lambda_n}$. For the second set the value of $\Theta_{nn}/2\sqrt{\lambda_n}$ tends to zero for increasing values of $\sqrt{\lambda_n}$. In Fig. 3(b) the relative influence (in per cent) of $\Theta_{nn}/2\lambda_n$ on $\sqrt{\lambda_n}$ is presented. From this figure it can be observed that $\Theta_{nn}/2\sqrt{\lambda_n}$ is relatively small compared to $\sqrt{\lambda_n}$. For the first set of frequencies this percentage tends to 0.7. Note that this value is exactly equal to $100r^2s^2/4$ for the case $r^2 = 0.01$ and $s^2 = 2.8$ (see also Eq. (124)).

5.1. An example

In this subsection the effect of gravity on the natural frequencies of a tall building will be examined. The building has a square cross-section, and the parameters of this building are given by $E = 25\times10^9\,\mathrm{N\,m^{-2}}$, $I = 2.5\times10^3\,\mathrm{m^4}$, $L = 180\,\mathrm{m}$, $\rho = 280\,\mathrm{kg\,m^{-3}}$, $A = 1225\,\mathrm{m^2}$, and $g = 9.81\,\mathrm{m\,s^{-2}}$. Moreover, $G = E/2(1+\nu)$ and $k = 5 + 5\nu/6 + 5\nu$, in which $\nu = 0.2$ is Poisson's ratio. Hence, the non-dimensional parameters r^2, s^2, and ε are given by 6.30×10^{-5}, 2.8, and 0.314, respectively. The building is modelled as a Timoshenko beam. Now the first ten natural frequencies (Ω_n) of the building are listed in Table 2. It can be observed from this table that the effect of gravity (σ_n) on the natural frequency (Ω_n) is largest for the first bending mode. There is a

Table 2
The effect of gravity (σ_n) on the first ten natural frequencies (Ω_n) of a tall building in Hertz (Hz) and in per cent (%) for the case $r^2 = 6.30 \times 10^{-5}$, $s^2 = 2.8$, and $\varepsilon = 0.314$

n	Ω_n (in Hz)	σ_n (in Hz)	$100\left(\frac{\sigma_n}{\Omega_n}\right)$ (in %)
1	0.2284	−0.00465	−2.0366
2	1.4513	−0.00409	−0.2818
3	4.0496	−0.00423	−0.1043
4	7.8792	−0.00446	−0.0567
5	12.903	−0.00463	−0.0359
6	19.055	−0.00476	−0.0250
7	26.265	−0.00488	−0.0186
8	34.455	−0.00506	−0.0147
9	53.467	−0.00035	−0.0007
10	64.130	−0.00052	−0.0001

reduction of 2.04% in the first natural frequency. For the other modes in Table 2 this effect is small, that is, smaller than 0.3%. In this section it has been shown that the effect of gravity increases by increasing mode number. For this tall building this is also the case. However, the effect of gravity will be relatively small compared to the magnitude of the natural frequency since the parameters r^2 and ε are small.

6. Conclusions

In this paper the transverse vibrations of a standing, cantilevered Timoshenko beam have been considered. Due to gravity and due to the self-weight of the beam a linear varying compression force is acting on the beam. It was assumed that the compression force is small but not negligible. Inclusion of the compression force into the beam model reduces the magnitude of the frequencies of the beam. In this paper this decrease in magnitude of the frequencies has been studied. Note that the results found in this paper can also be applied to hanging beams. In this case a linearly varying tensile force is acting on the beam. Inclusion of this force into the beam model results in an increase in the magnitude of the frequencies of the beam. In Ref. [8] the natural frequencies of a hanging beam under gravity has been studied. Here, it has been concluded that the influence of gravity on the frequencies of the hanging beam reduces by increasing mode number. In this paper similar results has been found for the lower order modes: the decrease in magnitude of the frequency due to the compression force will become relatively small (compared to the magnitude of the frequency) by increasing mode number. However, it also has been found that the frequencies of the higher order modes can be separated into two sets of frequencies. For the first set of frequencies it has been found that the decrease in magnitude of the frequency due to the compression force increases significantly by increasing mode number. Moreover, it has been concluded that the inclusion of shear deformation and rotatory inertia into the beam model increases the decrease in magnitude (due to the compression force) of the frequencies of a standing, cantilevered beam. And consequently, these inclusions into the beam model of a hanging beam results in an increase in magnitude of the frequencies. Note that this is different from the conclusion in Ref. [8], where it has been stated that these inclusions reduces the increases in the higher mode frequencies of the hanging beam due to gravity effects. For the second set of frequencies it has been concluded that the decrease in magnitude of the frequencies is less significant compared to the decrease in magnitude of the frequencies of the first set.

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